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Numerical Solution of Hyperbolic Systems with Different Time Scales Using Asymptotic Expansions

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For a hyperbolic system with different time scales, initial and boundary conditions can be given such that the fast time scale is not initiated. The solutions obtained can be developed in an asymptotic series. Computation of this series is done by a difference scheme, where the time step is not restricted by the fast time scale. The method is applied to the simplest version of the MHD equations.

1. INTRODUCTION

Many time-dependent processes in physics and technology are such that time scales of different magnitudes are present. If the mathematical model is a hyperbolic system of partial differential equations of first order, the different time scales are represented by the eigenvalues of the coefficient matrix, with the large eigenvalues corresponding to the fast time scale. In many applications the solutions consist of a slowly varying part perturbed by a small rapidly oscillating part. In such cases one is often interested in the unperturbed solution only. A typical example is the dynamics of the atmosphere in meteorology, where the fast gravity waves are small and of no influence on weather forecasting.

Another example comes from plasma physics, where the so-called fast waves of the MHD equations in some applications are much faster than the remaining wavetypes. If an explicit difference scheme is used for this system, and the primary interest is in the slow waves, then the computing time becomes too long because of the stability limit on the time step. Therefore some method must be used which permits the use of larger time steps. A fully implicit scheme (perhaps with a smoothing term included) is one solution to the problem, and is advocated as the best solution by Brackbill in his survey article [1]. One can also use a semi-implicit scheme; see, for example, [11]. Recently, a method using the grid as a dynamic variable has been proposed for the MHD equations by Jardin *et al* [9].

If an implicit scheme is used with a time step which is significantly longer than what would be allowed by the Courant–Friedrichs–Lewy condition, then we can expect accurate solutions only if the slow waves are the dominant part of the solution. To be

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more precise, we require that the derivatives of the solution up to some order p be bounded, independently of the magnitude of the large eigenvalues.

Recently, Kreiss [12, 13] has developed a general theory for problems with different time scales, and the main result is that p derivatives will be bounded during a certain time interval provided p derivatives are bounded initially. In other words, with proper initial data and boundary conditions the fast time scale is not initiated. The kind of problems we are considering here are characterized by a small parameter ϵ occuring in the system of differential equations. Kreiss' theory for ordinary differential equations makes use of asymptotic expansions in terms of ϵ; such expansions have been considered also by other authors, for example, in [6–8, 10]. In [5] we proved the existence of asymptotic expansions also for partial differential equations, under certain conditions on the initial data, such that the solutions are smooth independently of ϵ. The main idea of this paper is that these expansions should be used also for computational purposes. In other words, if we are interested in smooth solutions only, i.e., the cases where implicit schemes could be used, then we should use the smoothness property to make the computation efficient. We will show, with the one-dimensional MHD equations as an example, that this will simplify the computation considerably compared to those methods where the whole system is integrated in time. An obvious way is to solve the reduced problem obtained for $\epsilon = 0$, which corresponds to using the zero-order term in the expansion only. However, in many applications, this method does not give sufficient accuracy. We will show that higher-order terms can be included and the computational procedure will still be efficient. In this way the system can be reduced to a smaller subsystem which does not contain the fast waves, and therefore can be integrated using a simple explicit difference method.

In Section 2 we define the problem and the method in general; an analysis of the stability is also given both for the differential equations and the difference schemes. In Section 3 we define the MHD problem and the method applied to it, and in Section 4 the numerical experiments are presented.

2. Difference Methods Using Asymptotic Expansions

We will consider hyperbolic systems in one space dimension, which have the form

$$\begin{pmatrix} u^{\mathrm{I}} \\ u^{\mathrm{II}} \end{pmatrix}_t + \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & \epsilon^{-1}P_{22} \end{pmatrix} \begin{pmatrix} u^{\mathrm{I}} \\ u^{\mathrm{II}} \end{pmatrix} + \begin{pmatrix} F^{\mathrm{I}} \\ \epsilon^{-1}F^{\mathrm{II}} \end{pmatrix} = 0, \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0, \tag{2.1}$$

where

$$P_{ij} = A_{ij}\,\partial/\partial x + E_{ij}\,, \qquad i = 1, 2; j = 1, 2.$$

Here ϵ is a small parameter and u^{I}, u^{II} are vector functions of x and t with r and s components, respectively.

A_{ij} are matrices which may depend also on u^{I}, u^{II}. It is always assumed that A_{22}^{-1} is bounded independently of ϵ.

If the original form of a given system is not (2.1), then it is assumed that a change of variables is made such that form (2.1) is obtained. This procedure is carried out for the MHD problem in Section 3.

There are also initial conditions

$$u^{\mathrm{I}}(x, 0) = f^{\mathrm{I}}(x), \tag{2.2a}$$

$$u^{\mathrm{II}}(x, 0) = f^{\mathrm{II}}(x) \tag{2.2b}$$

and boundary conditions

$$B_0 u(0, t) = g_0(t), \tag{2.3a}$$

$$B_1 u(1, t) = g_1(t), \tag{2.3b}$$

where B_0, B_1 are rectangular matrices, and u is the vector $u = (u^{\mathrm{I}}, u^{\mathrm{II}})^T$. In [13, 5] conditions on the boundary conditions are given such that smooth solutions exist which can be developed into an asymptotic series. In Section 3 we will investigate these conditions for the MHD example. Let us just note here that the reduced problem corresponding to $\epsilon = 0$ must be well posed. As an example, consider the problem

$$u_t + \begin{pmatrix} a & 1 & 0 \\ 1 & a & 1/\epsilon \\ 0 & 1/\epsilon & a \end{pmatrix} u_x = 0, \qquad a > 0,$$

$$u(x, 0) = f(x),$$

$$u^{(1)}(0, t) = 0,$$

$$u^{(1)}(0, t) + bu^{(2)}(0, t) + cu^{(3)}(0, t) = 0,$$

$$u^{(1)}(1, t) = 0.$$

These are well-posed boundary conditions for any fixed $\epsilon > 0$. However, the reduced problem is

$$u_t^{(1)} + au_x^{(1)} = 0, \qquad u^{(1)}(x, 0) = f^{(1)}(x),$$

$$u_x^{(2)} = u_x^{(3)} = 0,$$

and in general there is obviously no unique solution to this problem. For a calculation made in practice with nonzero but small ϵ, we would get nonsmooth solutions.

We introduce the truncated formal asymptotic expansion for u^{II}

$$u^{\mathrm{II}} = \sum_{j=0}^{p-1} \epsilon^j \phi_j$$

into the differential equation. In general, the elements of A_{21}, A_{22}, E_{21}, E_{22}, and F^{II} are expanded in terms of ϵ, and all terms corresponding to each power of ϵ are collected and put equal to zero.

Assume that A_{21}, A_{22} for a given vector function u^{I} are smooth matrix functions of u^{II}, so that we have expansions

$$A_{21} = \sum_{j=0}^{p-1} A_{21}^{(j)} \epsilon^j,$$

$$A_{22} = \sum_{j=0}^{p-1} A_{22}^{(j)} \epsilon^j,$$

where $A_{21}^{(j)}$ and $A_{22}^{(j)}$ depend on ϕ_0, ϕ_1,..., ϕ_j only. In the same way we assume

$$F^{\mathrm{II}} = \sum_{j=0}^{p-1} F^{(j)} \epsilon^j, \qquad E_{2i} = \sum_{j=0}^{p-1} E_{2i}^{(j)} \epsilon^j, \qquad i = 1, 2,$$

Then the equations for ϕ_j are

$$\begin{aligned} &A_{22}^{(0)} \phi_{0_x} + E_{22}\phi_0 + F^{(0)} = 0, \\ &(\phi_{j-1})_t + A_{21}^{(j-1)} u_x{}^{\mathrm{I}} + E_{21}^{(j-1)} u^{\mathrm{I}} + \sum_{k=0}^{j-1} A_{22}^{(k)} (\phi_{j-k})_x \\ &\quad + E_{22}^{(k)} \phi_{j-k} + F^{(j)} = 0, \qquad j = 1, 2,..., p-1. \end{aligned} \tag{2.4}$$

For a given vector function u^{I} Eqs. (2.4) are solved using boundary conditions (2.3). The functions $\phi_j(x, 0)$ determine the initial data f^{II} up to order ϵ^{p-1} if we want bounded derivatives up to order p; see [5].

We note that if $E = 0$ and $F^{\mathrm{II}} = 0$, then $\phi_{0_x} = 0$ and apparently all systems in the second equations of (2.4) are linear in the unknown variable ϕ_j.

We also write down system (2.4) if all the coefficients A_{ij} and E_{ij} of (2.1) are constant and independent of ϵ:

$$\begin{aligned} &A_{22}\phi_{0_x} + E_{22}\phi_0 + F^{\mathrm{II}} = 0, \\ &\phi_{0_t} + A_{21} u_x{}^{\mathrm{I}} + E_{21} u^{\mathrm{I}} + A_{22}\phi_{1x} + E_{22}\phi_1 = 0, \\ &(\phi_{j-1})_t + A_{22}\phi_{j_x} = 0, \qquad j = 2, 3,..., p-1. \end{aligned} \tag{2.4a}$$

We now turn to the method of calculation. The problem at hand with the initial function determined is to compute u^{I} and $u^{\mathrm{II}} = \sum_{j=0}^{p-1} \epsilon^j \phi_j$ in an efficient way. The general approach is to use an explicit difference scheme for the first part of the system of differential equations. The time step is restricted essentially by the eigenvalues of A_{11} only, and this is of course the main point which makes the calculation efficient. We do not discuss here the various schemes which could be used, but the leap-frog scheme with second- or fourth-order accuracy in the x-direction, possibly with a dissipation term included, is one of the candidates; see [11, 14, 2]. For each time step in the general case, one or more systems of ordinary differential equations must be

solved. Sometimes this system can be solved once and for all independently of u^{I} with t as a parameter as in our MHD example. In other cases the system can be solved analytically for the given t-value at each time step.

Let us assume that the boundary conditions have the form

$$\begin{aligned} C_0 u^{\mathrm{I}}(0, t) &= h_0^{\mathrm{I}}(t), \\ C_1 u^{\mathrm{I}}(1, t) &= h_1^{\mathrm{I}}(t); \end{aligned} \tag{2.5}$$

$$\begin{aligned} u^{\mathrm{II}}(0, t) &= h_0^{\mathrm{II}}(t, u^{\mathrm{I}}(0, t)), \\ u^{\mathrm{II}}(1, t) &= h_1^{\mathrm{II}}(t, u^{\mathrm{I}}(1, t)). \end{aligned} \tag{2.6}$$

Here (2.6) represents s conditions; (2.2) represents r or fewer conditions such that $u_t + A_{11}u_x$ is well posed with (2.2a), (2.5).

The general algorithm for computing $u(t + \Delta t)$ when all time levels up to time t are known is divided into two main steps:

1. Advance the difference scheme for

$$u_t^{\mathrm{I}} + P_{11}u^{\mathrm{I}} + P_{12}u^{\mathrm{II}} + F^{\mathrm{I}} = 0$$

one step:

$$u^{\mathrm{I}^{n+1}} = \sum_j Q_j u^{n-j} - \tilde{F}^{\mathrm{I}^n}. \tag{2.7}$$

Here stable numerical boundary conditions must be used together with (2.5) such that $u^{\mathrm{I}}(t + \Delta t)$ is defined also at the boundaries.

2. Solve (2.4) with ϕ_0 satisfying (2.6) and $\phi_j = 0$ at $x = 0$, $x = 1, j = 1,\ldots, p - 1$. This is done analytically if possible, otherwise by a difference approximation

$$R_j\phi_j = 0, \qquad j = 0, 1,\ldots, p - 1. \tag{2.8}$$

Boundary conditions (2.6) could also be expanded in terms of ϵ, such that ϕ_j satisfies the jth-order part, but there is no disadvantage in letting ϕ_0 satisfy (2.6) exactly.

The time derivatives $(\phi_{j-1})_t$ occuring in (2.4) can sometimes be computed analytically, as in our MHD example. They can also be substituted by x-derivatives of u^{I} and already computed $\phi_j s$ by using the first part of the PDE system, but in general it is more convenient to use a difference approximation in time. This general method will be treated in the stability analysis below.

For practical purposes we believe that it is sufficient to consider the cases $p = 1, 2, 3$. For each case we will first show explicitly that the resulting system of differential equations is well posed. Furthermore the stability is analyzed for the corresponding approximation based on the second-order leap-frog scheme for (2.7) and centered second-order difference approximations for (2.8). Only the Cauchy problem for

symmetric constant coefficients is considered, where for the u^{II}-equations we require $u^{II}(\pm\infty) = 0$. The lower-order terms in (2.4a) are disregarded, since they do not affect the stability.

$p = 1$

This is the reduced problem, and we have

$$u_x^{II} = \phi_{0_x} = 0.$$

Therefore

$$u_t^{I} + A_{11}u_x^{I} = 0,$$

and since A_{11} is symmetric we get with

$$(u, v) = \int u^T v \, dx, \qquad \| u \|^2 = (u, u),$$

that

$$\frac{d}{dt} \| u^{I} \|^2 = -2(u^{I}, A_{11}u_x^{I}) = 0.$$

This leads immediately to

$$\| u(t) \| = \| u(0) \| .$$

The approximation is

$$\frac{u^{I^{n+1}} - u^{I^{n-1}}}{2\Delta t} + A_{11}D_0 u^{I^n} + A_{12}D_0 u^{II^n} = 0,$$

where D_0 is the centered second-order difference operator. Since $D_0 u^{II^n} = D_0 \phi_0{}^n = 0$, stability is immediately clear for $| \lambda |_{\max} \Delta t < \Delta x$, where the eigenvalues of A_{11} satisfy $| \lambda_j | \leqslant | \lambda |_{\max}$; see, for example, [15].

$p = 2$

In this case we have

$$u_x^{II} = \epsilon \phi_{1_x} = -\epsilon A_{22}^{-1} A_{21} u_x^{I},$$

and therefore

$$u_t^{I} + A_{11}u_x^{I} - \epsilon A_{12}A_{22}^{-1}A_{21}u_x^{I} = 0.$$

A_{11} and A_{22} are symmetric and $A_{12}^T = A_{21}$. Hence, the matrix $A_{11} - \epsilon A_{12}A_{22}^{-1}A_{21}$ is symmetric, and the estimate

$$\| u(t) \| = \| u(0) \|$$

follows as in the previous case.

The approximation for $u^{II} = \epsilon\phi_1$ satisfies

$$A_{22}D_0\phi_1{}^n = -\epsilon A_{21}D_0u^{I^n},$$

and we get

$$\frac{u^{I^{n+1}} - u^{I^{n-1}}}{2\Delta t} + A_{11}D_0u^{I^n} - \epsilon A_{12}A_{22}^{-1}A_{21}D_0u^{I^n} = 0.$$

The stability condition is slightly strengthened; we must require that $|\tilde{\lambda}|_{\max}\Delta t < \Delta x$, where the eigenvalues $\tilde{\lambda}_j$ of $A_{11} - \epsilon A_{12}A_{22}^{-1}A_{21}$ satisfy $|\tilde{\lambda}_j| \leqslant |\tilde{\lambda}|_{\max}$.

$p = 3$

u^{II} now has the form

$$u^{II} = \epsilon\phi_1 + \epsilon^2\phi_2,$$

where ϕ_2 is defined by

$$\phi_{2_x} = -A_{22}^{-1}\phi_{1_t} = -A_{22}^{-2}A_{21}u_t{}^I\epsilon.$$

By inserting the expression for u^{II} into the first part of the system we get the system for u^I

$$Cu_t{}^I + Du_x{}^I = 0,$$

where

$$C = I - \epsilon^2A_{12}A_{22}^{-2}A_{21}$$

and

$$D = A_{11} - \epsilon A_{12}A_{22}^{-1}A_{21}$$

are symmetric matrices. C is positive definite and therefore we can define a new norm (u, Cu), and obtain

$$\frac{d}{dt}(u, Cu) = -2(u, Du_x) = 0.$$

Hence

$$\begin{aligned}\|u(t)\|^2 \leqslant K_1(u(t), Cu(t)) &= K_1(u(0), Cu(0)) \\ &\leqslant K_2\|u(0)\|^2.\end{aligned}$$

The numerical calculation of ϕ_2 requires in general an approximation of ϕ_{1_t}. Since $\epsilon^2\phi_2$ is the last term in the expansion, it does not make sense to require too high accuracy; therefore we use a one-sided difference approximation:

$$A_{22}D_0\phi_2^{n+1} + \frac{\phi_1^{n+1} - \phi_1{}^n}{\Delta t} = 0.$$

For u^I we get

$$\frac{u^{I^{n+1}} - u^{I^{n-1}}}{2\Delta t} + A_{11}D_0u^{I^n} - \epsilon A_{12}A_{22}^{-1}A_{21}D_0u^{I^n} - \epsilon^2A_{12}A_{22}^{-2}A_{21}\frac{u^{I^n} - u^{I^{n-1}}}{\Delta t} = 0.$$

Because of the last term the amplification factor will have a magnitude $1 + O(\epsilon^2)$. However, the increase in the solution will be of no practical importance, since at time t the truncation error and the rounding errors are multiplied by a factor $e^{O(\epsilon^2)t/\Delta t}$, which for $\Delta t = O(\epsilon)$, say, is of order $e^{O(\epsilon t)}$.

If one insists upon having no growing component of the solution, the equation for computing ϕ_2^{n+1} is changed to

$$A_{22}D_0\phi_2^{n+1} + \frac{\phi_1^{n+2} - \phi_1^{n}}{2\Delta t} = 0.$$

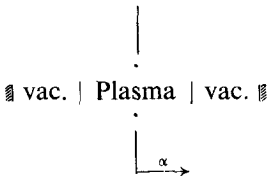
Since ϕ_1^{n+2} is in general not available, we introduce the expression for ϕ_2 into the equation for u^{I} and obtain

$$(I - \epsilon^2 A_{12}A_{22}^{-2}A_{21})\frac{u^{\mathrm{I}^{n+1}} - u^{\mathrm{I}^{n-1}}}{2\Delta t} + A_{11}D_0u^{\mathrm{I}^n} - \epsilon A_{12}A_{22}^{-1}A_{21}D_0u^{\mathrm{I}^n} = 0,$$

which is an approximation of the differential equation $Cu_t^{\mathrm{I}} + Du_x^{\mathrm{I}} = 0$ above. The step size is now determined by the eigenvalues of $(\mathrm{I} - \epsilon^2 A_{12}A_{22}^{-2}A_{21})^{-1}(A_{11} - \epsilon A_{12}A_{22}^{-1}A_{21})$.

3. The MHD Equations

For illustration and in order to test the method we have proposed, the computational procedure was applied to a simple one-dimensional problem from plasma physics. It describes a plasma surrounded by vacuum which is confined between two walls. The equations are



$$\begin{aligned} &\rho_t + v\rho_x + \rho v_x = 0, \\ &v_t + \frac{a^2}{p}\rho_x + vv_x + \frac{B}{4\pi\mu\rho}B_x = 0, \\ &B_t + Bv_x + vB_x = 0, \qquad 0 \leqslant x \leqslant \alpha(t), \end{aligned} \tag{3.1}$$

where the dependent variables are ρ = density, v = velocity, B = magnetic induction field. a is the speed of sound, μ is the permeability. We have assumed constant entropy, and that the pressure is defined by $p = Ap^{\gamma}$, where A and γ are given constants. In

this way the speed of sound is defined by $a^2 = A\gamma\rho^{\gamma-1}$. $\alpha(t)$ is the distance between the center and the plasma/vacuum interface. It is defined by

$$\alpha(t) = \int_0^t v(1, t)\, dt + \alpha_0 ,$$

where α_0 is the initial radius.

The boundary conditions are

$$v = 0 \qquad \text{at } x = 0, \tag{3.2a}$$

$$8\pi\mu A p^{\gamma} + B^2 = g(t) \qquad \text{at } x = \alpha. \tag{3.2b}$$

Here (3.2b) denotes the fact that the total pressure is continuous across the plasma–vacuum interface; the magnetic field in vacuum is a given function of time.

The variables are made dimensionless by

$$\begin{aligned} \hat{\rho} &= \rho/\rho_0 , \\ \hat{v} &= v/v_0 , \\ \hat{B} &= B/B_0 , \\ \hat{x} &= x/\alpha, \\ \hat{t} &= tv_0/\alpha_0 , \end{aligned}$$

from which we get the new system

$$\begin{aligned} &\rho_t + \frac{\alpha_0}{\alpha}\left[(v - v(1, t)x)\,\rho_x + \rho v_x\right] = 0, \\ &v_t + \frac{\alpha_0}{\alpha}\left[\frac{1}{M_0^2\rho^{2-\gamma}}\,\rho_x + (v - v(1, t)x)\,v_x + \frac{B}{\epsilon^2\rho}\,B_x\right] = 0, \\ &B_t + \frac{\alpha_0}{\alpha}\left[Bv_x + (v - v(1, t)x)\,B_x\right] = 0, \\ &\alpha = \alpha_0 \int_0^t v(1, t)\, dt + \alpha_0 . \end{aligned} \tag{3.3}$$

Here we have dropped the "hats," so that x, t, ρ, v, B are the new variables. α is a function of t; therefore differentiation with respect to time produces an extra term $\alpha'(t)\,\hat{x}(\partial/\partial x) = v(1, t)\,\hat{x}(\partial/\partial x)$, and this is the reason for the modified diagonal coefficients. ϵ is a small parameter representing the inverse of the Alfvén number, and is defined by

$$\epsilon = \frac{v_0}{B_0}(4\pi\mu\rho_0)^{1/2}.$$

The Mach number M_0 is defined by

$$M_0 = v_0/a_0 = v_0/(A\gamma\rho_0^{\gamma-1})^{1/2}.$$

The coefficient matrix for the space derivative is with $\tilde{v} = v - v(1, t)\, x$

$$A = \frac{\alpha_0}{\alpha} \begin{pmatrix} \tilde{v} & \rho & 0 \\ \dfrac{1}{M_0^2 \rho^{2-\gamma}} & \tilde{v} & \dfrac{B}{\epsilon^2 \rho} \\ 0 & B & \tilde{v} \end{pmatrix},$$

which has the eigenvalues

$$\lambda_1 = \frac{\alpha_0}{\alpha} \tilde{v},$$

$$\lambda_{2,3} = \frac{\alpha_0}{\alpha} \left(\tilde{v} \pm \left(\frac{\rho^{\gamma-1}}{M_0^2} + \frac{B^2}{\epsilon^2 \rho} \right)^{1/2} \right).$$

We introduce the new variable $\tilde{B}$ by $B = 1 + \epsilon \tilde{B}$ and get the new system

$$w_t + A(w)\, w_x = 0, \qquad 0 \leqslant x \leqslant 1, \tag{3.4}$$

where $w = (\rho, v, \tilde{B})^T$,

$$A(w) = \frac{\alpha_0}{\alpha} \begin{pmatrix} \tilde{v} & \rho & 0 \\ \dfrac{1}{M_0^2 \rho^{2-\gamma}} & \tilde{v} & \dfrac{B}{\epsilon \rho} \\ 0 & \dfrac{B}{\epsilon} & \tilde{v} \end{pmatrix}.$$

The boundary conditions are

$$v = 0 \qquad \text{at } x = 0, \tag{3.5a}$$

$$2\epsilon^2 \rho_0^{\gamma-1} \frac{A}{v_0^2} \rho^\gamma + (1 + \epsilon \tilde{B})^2 = \frac{g(t)}{B_0^2} = 1 + \epsilon \tilde{g}(t) \qquad \text{at } x = 1. \tag{3.5b}$$

At this point we check the conditions stated in [5] for the linearized version of (3.4), (3.5) with constant coefficients, such that the asymptotic expansion exist. We put $\tilde{v} = 0$ in $A(w)$ since $\tilde{v}$ vanishes at both boundaries. A simple scaling of the variables

$$v \to M_0 \hat{\rho}^{(3-\gamma)/2}\, v,$$
$$\tilde{B} \to M_0 \hat{\rho}^{(2-\gamma)/2} \tilde{B},$$

where $\hat{\rho}$ denotes the constant value of ρ in $A(w)$, gives $A(w)$ a symmetric form $\hat{A}$. The orthogonal matrix T containing the eigenvectors of $\hat{A}$ can be shown to have the form

$$T = \begin{pmatrix} t_{11} & \epsilon t_{12} & \epsilon t_{12} \\ 0 & t_{22} & -t_{22} \\ t_{31} & t_{32} & t_{32} \end{pmatrix},$$

where the $t_{ij}s$ are bounded for small ϵ-values, and $T^T\hat{A}T = \Lambda$ is diagonal. The transformed system is with $W = T^T w = (W_1, W_2, W_3)$

$$W_t + \frac{\hat{\alpha}_0}{\hat{\alpha}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & c/\epsilon & 0 \\ 0 & 0 & -c/\epsilon \end{pmatrix} W_x = 0, \qquad c > 0.$$

The linearized homogeneous boundary conditions expressed in the new variables have the form

$$\begin{aligned} W_2 &= W_3 && \text{at } x = 0, \\ W_3 &= -W_2 + b\epsilon W_1 && \text{at } x = 1, \end{aligned}$$

but since $W_1(1, t) = W_1(1, 0)$ it is sufficient to consider the condition

$$W_3 = -W_2 \qquad \text{at } x = 1.$$

The inequality required in [5] leading to an energy estimate can be expressed as

$$\text{Real}(W, (\hat{\alpha}_0/\hat{\alpha})\Lambda W_x) \leqslant -\delta(|W_2(0, t)|^2 + |W_3(1, t)|^2 + C\|W\|^2,$$

where $(\cdot, \cdot)$ denotes some scalar product, and $\|\cdot\|$ the corresponding norm.

We define for real functions

$$(W^{(1)}, W^{(2)}) = \int_0^1 W^{(1)^T} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \theta(x) \end{pmatrix} W^{(2)}\, dx,$$

where $\theta(x)$ is a smooth function. We get

$$(W, \Lambda W_x) = \frac{c}{\epsilon}[W_2(0, t)^2 - W_2(1, t)^2 - \theta(0)\, W_3(0, t)^2 + \theta(1)\, W_3(1, t)^2]$$

$$- \frac{c}{\epsilon}\int_0^1 W_3 \theta W_{3_x}\, dx - \frac{c}{\epsilon}\int_0^1 \theta_x W_3^2\, dx.$$

By choosing $\theta(0) = 1 + \epsilon$, $\theta(1) = 1 - \epsilon$, we can construct $\theta(x)$ such that $|\theta_x(x)| = O(\epsilon)$, and the required inequality is obtained by using the boundary conditions.

For the original problem (3.4), (3.5) it is now assumed that the time derivatives of v and B up to order p are bounded for small ϵ. The formal expansions

$$\begin{aligned} v &= \sum_{j=0}^{p-1} v_j \epsilon^j, \\ \tilde{B} &= \sum_{j=0}^{p-1} \tilde{B}_j \epsilon^j \end{aligned} \tag{3.6}$$

are inserted into the second and third differential equation, all terms with the same power of ϵ are collected, and the sum is put equal to zero. For a given function ρ the resulting equations are solved using for v the boundary conditions

$$v_j(0, t) = 0, \qquad j = 0, 1, \ldots, p - 1.$$

For $\tilde{B}$ the boundary conditions are obtained from the expansion of the condition

$$\tilde{B}(1, t) = (1/\epsilon)((1 + \epsilon\tilde{g}(t) - 2\epsilon^2\rho_0 A\rho(1, t)^\gamma/v_0^2)^{1/2} - 1) \equiv \Phi(t), \tag{3.7}$$

where the first terms are

$$\tilde{B}_0(1, t) = \frac{\tilde{g}(t)}{2},$$

$$\tilde{B}_1(1, t) = -\frac{\tilde{g}(t)^2}{8} + \rho(1, t)^\gamma \frac{\rho_0 A}{v_0^2}.$$

Since $v(x, t) = O(\epsilon)$, it is seen from the first equation in (3.3) that $\rho_t = O(\epsilon)$, and therefore we have at the boundary

$$\tilde{B}_{1_t}(1, t) = -\tilde{g}(t)\, g'(t)/4.$$

The first functions in the expansions are now easily obtained.

$$v_0(x, t) = 0,$$
$$\tilde{B}_0(x, t) = \tilde{g}(t)/2,$$
$$v_1(x, t) = -\frac{\alpha(t)}{\alpha_0} \cdot \frac{\tilde{g}'(t)}{2} x,$$
$$\tilde{B}_1(x, t) = \frac{1}{\gamma M_0^2}(\rho(1, t)^\gamma - \rho(x, t)^\gamma) - \frac{\tilde{g}(t)^2}{8} + \rho(1, t)^\gamma \frac{\rho_0 A}{v_0^2},$$
$$v_2(x, t) = \frac{\alpha(t)}{\alpha_0} \cdot \frac{\tilde{g}(t)\,\tilde{g}'(t)}{2} x.$$

The initial conditions are restricted such that (3.6) is satsified for $t = 0$. We could add terms of order ϵ^p to these expansions, but the essential freedom in the choice of initial data is that $\rho(x, 0)$ can be chosen arbitrarily.

We will now discuss the solutions to the full system in some detail, depending on the number of terms in the expansions.

1. $v = v_0$, $\tilde{B} = \tilde{B}_0$

This case is equivalent to setting $\epsilon = 0$ in the original system. The assumption is that the first-order time derivatives are bounded, and apparently this requires that $\tilde{g}'(t)$ be bounded. The initial conditions are

$$v(x, 0) = O(\epsilon),$$
$$\tilde{B}(x, 0) = \tilde{g}(0)/2 + O(\epsilon).$$

The solution in this case is trivial and has an error of order ϵ:

$$\begin{aligned} \rho(x, t) &= \rho(x, 0), \\ v(x, t) &= 0, \\ \tilde{B}(x, t) &= \tilde{g}(t)/2, \\ \alpha(t) &= \alpha_0 . \end{aligned} \tag{3.8}$$

Obviously, solving the reduced problem does not give any detailed information about the solution except if ϵ is very small.

2. $v = v_0 + \epsilon v_1 , \tilde{B} = \tilde{B}_0 + \epsilon \tilde{B}_1$

In this case the time derivatives of order 2 are assumed to be bounded. This requires

[redacted]

$$v(x, 0) = -0.5\epsilon \tilde{g}(0)x + O(\epsilon^2),$$

$$\tilde{B}(x, 0) = \frac{\tilde{g}(0)}{2} + \epsilon \left(\frac{1}{\gamma {M_0}^2} (\rho(1, 0)^\gamma - \rho(x, 0)^\gamma) - \frac{\tilde{g}(0)^2}{8} + \rho(1, 0)^\gamma \frac{\rho_0 A}{{v_0}^2} \right) + O(\epsilon^2).$$

Since v is a linear function of x, the first equation of the system takes the simple form

$$\rho_t - \frac{g'}{2} \epsilon \rho = 0.$$

The equation for α is

$$\frac{d\alpha}{dt} = -\frac{\epsilon}{2\alpha_0} \tilde{g}' \alpha,$$

and we can easily write down the complete solution:

$$\begin{aligned} \rho(x, t) &= e^{(\epsilon/2)(\tilde{g}(t) - \tilde{g}(0))} \rho(x, 0), \\ v(x, t) &= -e^{-(\epsilon/2\alpha_0)(\tilde{g}(t) - \tilde{g}(0))} \tfrac{1}{2} \tilde{g}'(t) \epsilon x, \\ \tilde{B}(x, t) &= \frac{\tilde{g}(t)}{2} + \epsilon \left[\frac{1}{\gamma {M_0}^2} (\rho(1, t)^\gamma - \rho(x, t)^\gamma) - \frac{\tilde{g}(t)^2}{2} \right] + \beta, \\ \alpha(t) &= e^{-(\epsilon/2\alpha_0)(\tilde{g}(t) - \tilde{g}(0))} \alpha_0 \end{aligned} \tag{3.9}$$

(the constant β is in the numerical experiments chosen such that the boundary condition (3.7) is satisfied exactly).

If we are interested in raising the accuracy for v only we can add the v_2-term:

$$\begin{aligned} v(x, t) &= -\frac{1}{2} \tilde{g}'(t) \epsilon x (1 - \epsilon \tilde{g}(t)) \frac{\alpha(t)}{\alpha_0}, \\ \alpha(t) &= e^{-(\epsilon/2\alpha_0)(\tilde{g}(t) - \tilde{g}(0)) + (\epsilon^2/4\alpha_0)(\tilde{g}(t)^2 - \tilde{g}(0)^2)} \alpha_0 . \end{aligned} \tag{3.10}$$

4. Numerical Results

In this section we present results from a number of numerical experiments with system (3.1).

These experiments are designed to show two things:

(1) The importance of initialization. Implicit schemes with large time steps work properly only if the initial data are chosen such that the time derivatives are bounded (the fast scale not initiated). This is shown in experiments A, B, and C.

(2) The scheme of type (2.7), (2.8) using asymptotic expansions computes the solution efficiently if the fast scale is not present. This is shown in experiments C, D, and E.

The problem is first solved by integrating the *complete* system using an explicit difference scheme with a fine mesh. In that way we obtain an almost exact solution to use for comparison. An implicit method is then used for the complete system with the time step chosen to fit the slow time scale, which is, in our example, controlled by the boundary data. It is shown that the implicit scheme gives erroneous results if the initial conditions are not chosen properly. This is not very sensational, but we also show that it is not sufficient to choose these conditions such that the reduced problem is satisfied initially ($v_x(x, 0) = \tilde{B}_x(x, 0) = 0$).

The proposed method, using asymptotic expansions with the first few terms included only, is for our example very simple to use, and the solutions can be computed analytically, which has been done in Section 3.

However, in order to study the properties of the discrete method, we ran a scheme based on the leap-frog scheme corresponding to (2.7) and the trapezoidal rule corresponding to (2.8). In order to get a case where the first two terms in the expansion cannot be obtained analytically, we also ran a case where Eq. (3.4) was modified by a forcing function:

$$w_t + A(w)\, w_x + F = 0, \qquad F = (F_1, \epsilon^{-1}F_2, \epsilon^{-1}F_3)^T. \tag{4.1}$$

The scheme is:

$$\begin{gathered}
\frac{\rho_j^{n+1} - \rho_j^{n-1}}{2\Delta t} + \frac{\alpha_0}{\alpha^n}\left(\tilde{v}_j^{n+1} D_0 \rho_j{}^n + \rho_j{}^n D_0 v_j^n\right) + F_{1_j} = 0, \\
D_- \tilde{B}_{0_j}^{n+1} + \frac{\tilde{\alpha}^{n+1}}{\alpha_0}\rho^{n+1} F_{2_j}^{n+1} = 0, \qquad \tilde{B}_{0_N}^{n+1} = \Phi^{n+1}, \\
D_+ v_0^{n+1} + \frac{\tilde{\alpha}^{n+1}}{\alpha_0} F_{3_j}^{n+1} = 0, \qquad v_0^{n+1} = 0, \\
D_-\hat{\tilde{B}}_{1_j} + \hat{\rho}\left[\frac{\hat{\alpha}}{\alpha_0}\frac{v_{0_j}^{n+1} - v_{0_j}^n + v_{0_{j-1}}^{n+1} - v_{0_{j-1}}^n}{2\Delta t} + \frac{1}{M_0^2 \hat{\rho}^{2-\gamma}} D_-\hat{\rho}_j + \hat{v}_0 D_- \hat{v}_0\right. \\
\left. + \frac{1}{\hat{\rho}_j} D_-\hat{\tilde{B}}_1 + \frac{\hat{\tilde{B}}_{0_j}}{\hat{\rho}_j} D_- \tilde{B}_{0_j}\right] = 0, \qquad \tilde{B}_{1_N}^{n+1} = 0,
\end{gathered} \tag{4.2}$$

$$D_+\hat{v}_{1_j} + \frac{\hat{\alpha}}{\alpha_0}\frac{\tilde{B}^{n+1}_{0_j} - \tilde{B}^{n}_{0_j} + \tilde{B}^{n+1}_{0_{j+1}} - \tilde{B}^{n}_{0_{j+1}}}{2\Delta t} + \hat{\tilde{B}}_{0_j} D_+\hat{v}_{0_j} + \hat{\tilde{v}}_{0_j} D_+\hat{\tilde{B}}_{0_j} = 0, \qquad v^{n+1}_{1_0} = 0,$$

$$v^{n+1} = v^{n+1}_0 + \epsilon v^{n+1}_1, \qquad \tilde{B}^{n+1} = \tilde{B}^{n+1}_0 + \epsilon \tilde{B}^{n+1}_1,$$

$$\frac{\alpha^{n+1} - \alpha^n}{\Delta t} = \alpha_0 \hat{\tilde{v}}_N .$$

Here the caret over a variable denotes an average over the time levels t^n and t^{n+1}, $\tilde{\alpha}^{n+1}$ is defined by linear extrapolation using the previous time levels. D_+, D_- denote the forward and backward divided difference operators, respectively. One-sided second-order difference operators were used for v in the equation for ρ_N.
The results obtained by this scheme are labeled A-E-d. The analytically obtained results for the case $F \equiv 0$ are labeled A-E-a.

Throughout the experiments the following values on the physical constants were used:

$$v_0 = 10^5,$$
$$B_0 = 23,$$
$$\alpha_0 = 1,$$
$$\gamma = 1.67,$$
$$A = 5.553 \cdot 10^{14},$$
$$\mu = 10^{-4}$$

Two different values for the reference density ρ_0 were used:

$$\rho_0 = 1.67 \cdot 10^{-7}$$

giving the dimensionless constants

$$M_0 = 0.61202, \qquad \epsilon = 0.062985,$$

and

$$\rho_0 = 1.67 \cdot 10^{-9}$$

giving

$$M_0 = 2.6042, \qquad \epsilon = 0.0062985.$$

To calculate the solutions numerically for the full system we have used the DCG system, described in [3], to generate the explicit leap-frog scheme and the implicit Crank–Nicholson scheme.

The DCG system in its present form does not accept symmetry conditions leading to derivative boundary conditions for hyperbolic systems. Therefore the problem was defined on the interval $-1 \leqslant x \leqslant 1$ with symmetric initial functions for ρ, $\tilde{B}$ and antisymmetric initial velocity. The numerical boundary conditions can be chosen by

DCG such that the outgoing characteristic variables are extrapolated using also previous time levels. For our application these conditions have a poor accuracy and give rise to an initial jump in the solution. Since it is of interest in more general cases, we will explain this behavior.

One of the characteristics corresponds to the eigenvalue $(v(x, t) - v(1, t)\, x)\, \alpha/\alpha_0$, which is zero at the boundaries. Therefore the appropriate model equation is

$$u_t + a(x)\, u_x = 0, \qquad a(0) = 0,$$
$$u(x, 0) = f(x),$$

for which we will study the behavior near the boundary $x = 0$. The first step is computed by

$$u_j^1 = f_j + \frac{\Delta t}{2\Delta x}\, a_j (f_{j+1} - f_{j-1}), \qquad j = 1, 2, \ldots,$$
$$u_0^1 = f_0 .$$

All later steps are computed using the leap-frog scheme with the stable boundary condition

$$u_0^{n+1} = 2u_1^n - u_2^{n-1}.$$

The local truncation error for this condition is

$$u(0, t^{n+1}) - u_0^{n+1} = (\Delta x^2 + \Delta t^2)\, u_{\xi\xi}(0, t^{n+1}) + O((\Delta x^2 + \Delta t^2)^{3/2}),$$

where ξ is the coordinate along the lines $t + (\Delta t/\Delta x)\, x = \text{const}$ going through the meshpoints. For a system like the one we are considering in this paper, the mesh is chosen such that $|\lambda|_{\max} \Delta t/\Delta x < 1$, where $|\lambda|_{\max}$ is much larger than $\max_x |a(x)|$. Therefore, $u_{\xi\xi}(0, t)$ is not small even if $a(0)$ is zero and the error becomes quite large compared to the error at inner points. However, for the first step the error at the boundary is zero; u_0^1 equals f_0, which is the correct value. The typical result with these boundary conditions is therefore a jump after the first step and then a smooth-looking solution. In Fig. F1 the density at the boundary is shown as a function of time for the case described in *D*, but with $\rho_0 = 1.67 \cdot 10^{-7}$.

In this figure we also show the result when the extrapolation procedure is replaced by the use of one-sided difference operators at the boundary. For our model example we obtain

$$u_0^{n+1} = u_0^{n-1} + 2a_0 \frac{\Delta t}{\Delta x}\, (u_1^n - u_0^n),$$

which leads to an unstable scheme for scalar equations and general functions $a(x)$; see [4]. However, in our case we have $a_0 = 0$, and also taking the first step into consideration the above condition reduces to $u_0^{n+1} = u_0^n$, and we get the exact solution at the boundary for all times.

In our problem one-sided differences were used for the first two equations, and together with the physical boundary condition this defines the complete solution at the boundary. In [2] one-sided differences were used for a similar system.

In all the experiments $g(t)$ has the form

$$g(t) = g_0 \cdot G(t), \qquad G(0) = 1,$$
$$g_0 = (2\epsilon^2 \rho_0^{\gamma-1} A / v_0^2 + 1)\, B_0^2.$$

In this way the conditions $\rho(1, 0) = 1, \tilde{B}(1, 0) = 0$ are compatible with boundary condition (3.5b).

The initial values for the density are

$$\rho(x, 0) = 0,9 + 0.1 \cos 2\pi x$$

in all cases.

To limit the number of figures, we have chosen to present the boundary location α as a function of t, and the velocity v as a function of x.

The result of the leap-frog run representing the exact solution is represented in all figures by a solid line. The time step is chosen automatically by the DCG system to satisfy the stability criterion. In cases A, B, C, $\Delta t \approx 0.004$ was used, in case D, $\Delta t \approx 0.0004$. The following experiments were performed.

A

$$\rho_0 = 1.67 \cdot 10^{-7}\ (\epsilon = 0.062985),$$
$$g(t) = g_0[1.05 + 0.05 \sin(0.2\pi t - \pi/2)] \qquad \text{for } t \leqslant 1,$$
$$v(x, 0) = 0.2 \sin 2\pi x,$$
$$\tilde{B}(x, 0) = 0.2(1 - \cos 2\pi x),$$
$$\Delta x = 0.1.$$

The initial data do not satisfy the reduced problem, and the solution is nonsmooth. As could be expected, the Crank–Nicholson solution with $\Delta t = 0.05$ is very poor also for the integrated quantity α; Figs. A1, A3. With the shorter time step $\Delta t = 0.025$ the result is slightly better for α, but v is completely wrong; Figs. A2, A3. Also looked upon as a function of t, v has large errors; at $x = 1$ this is seen by studying the derivative of α in Fig. A2.

B

As in A but with

$$v(x, 0) = 0,$$
$$\tilde{B}(x, 0) = 0,$$
$$\Delta x = 0.05.$$

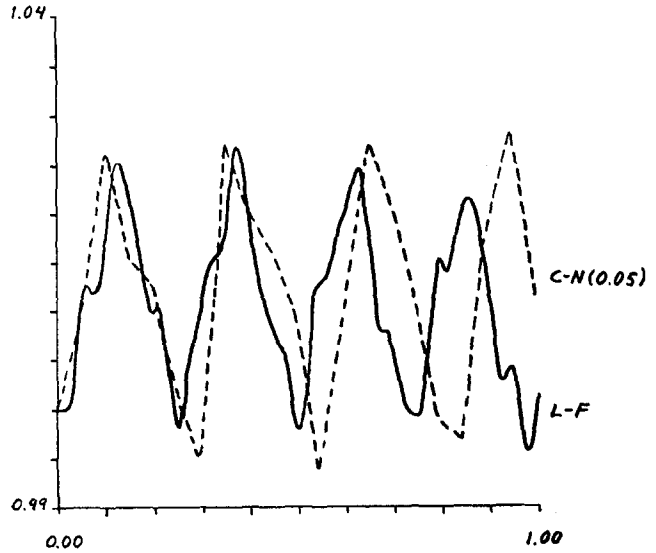


FIG. A1. $\alpha(t)$ for case A computed by the leap-frog scheme (L–F), and by the Crank–Nicholson scheme with $\Delta t = 0.05$(C–N(0.05)).

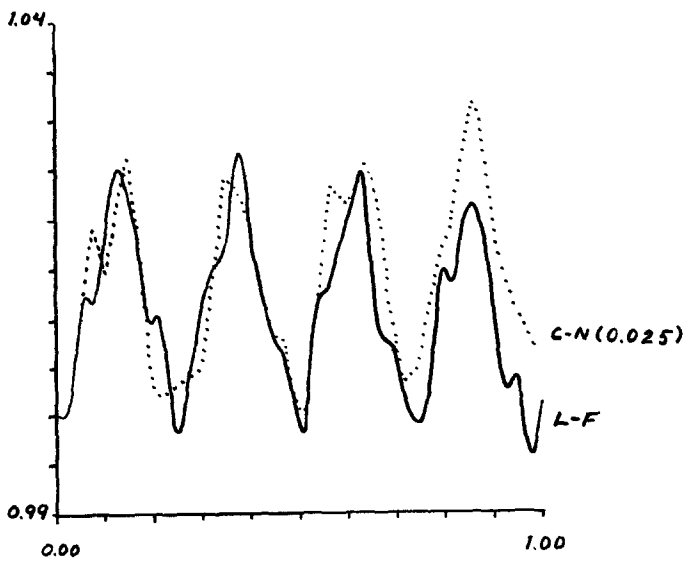


FIG. A2. $\alpha(t)$ for case A. L–F and C–N(0.025).

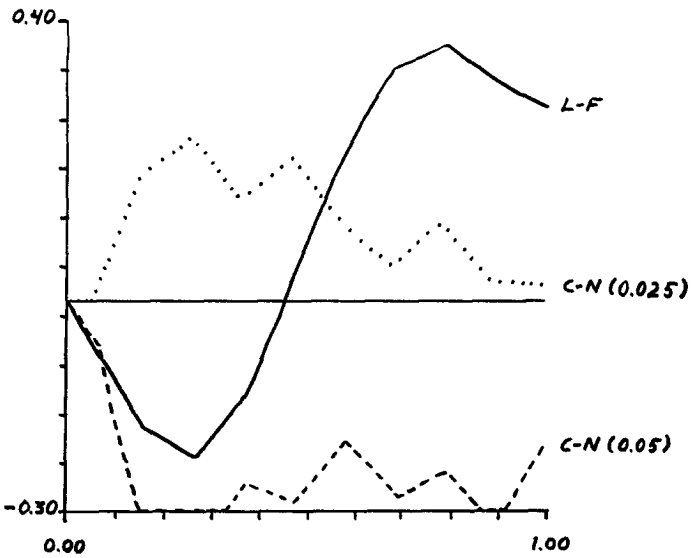
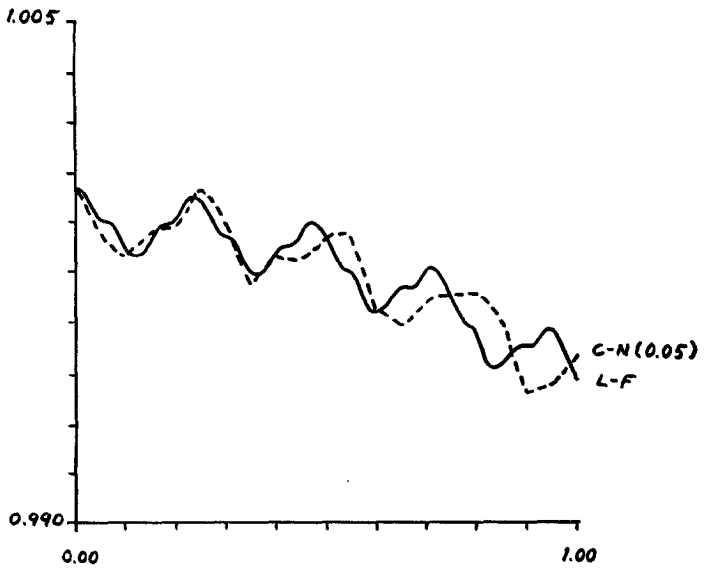


FIG. A3. $v(x, 1)$ for case A. L–F, C–N(0.05) and C–N(0.025).



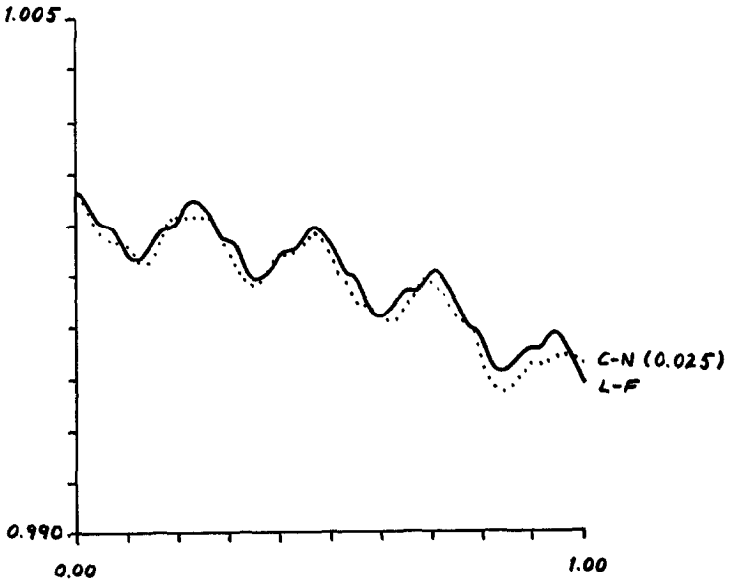


FIG. B2. $\alpha(t)$ for case B. L–F and C–N(0.025).

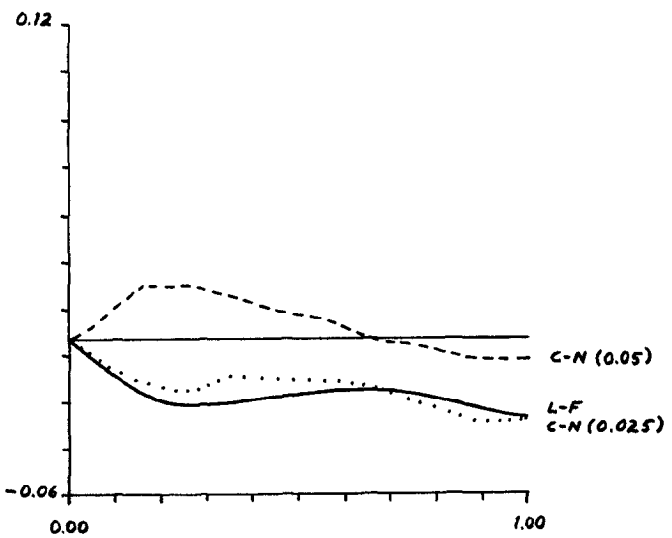


FIG. B3. $v(x, 1)$ for case B. L–F, C–N(0.5) and C–N(0.025).

The initial data satisfy the reduced problem. The Crank–Nicholson v-solutions are still very poor approximations as can be seen in Fig. B3. The α-solution using $\Delta t = 0.05$ is also poor; with the smaller time step the error is smaller but still noticeable.

C

As in B but with

$$\tilde{B}(x, 0) = [\rho(1, 0)^\gamma - \rho(x, 0)^\gamma]\, \epsilon/\gamma M^2$$
$$= [1 - (0.9 + 0.1 \cos 2\pi x)^\gamma]\, \epsilon/\gamma M^2.$$

These initial data guarantee that the time derivatives up to second order are bounded. The Crank–Nicholson solutions are quite accurate and cannot be distinguised from the true solution in Fig. C1. The oscillations in v for the Crank–Nicholson solution are small in absolute magnitude, and are due to the fact that the initial values are defined analytically, not by using the difference scheme.

The solution obtained from the asymptotic expansions is very accurate and cannot be distinguished in the figure for α. We obtain

$\alpha(1) = 0.99518$ when using (3.9), A-E-a,
$\alpha(1) = 0.99517$ when using (3.9), A-E-D,
$\alpha(1) = 0.99526$ when using (3.10), A-E-a.

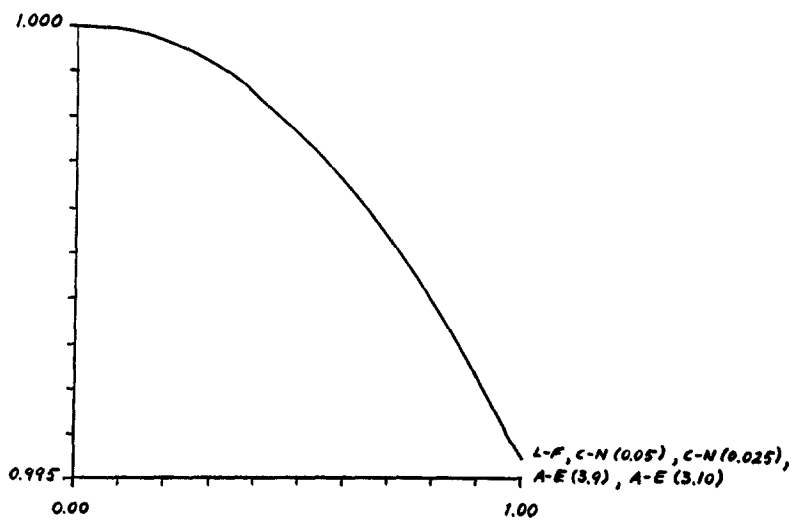


FIG. C1. $\alpha(t)$ for case C. L–F and C–N(0.05).

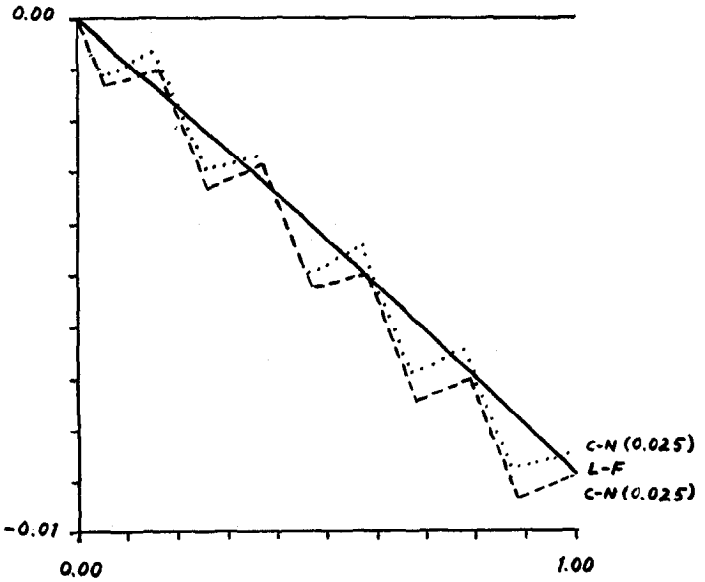


FIG. C2. $v(x, 1)$ for case C using L–F, C–N(0.05), and C–N(0.025).

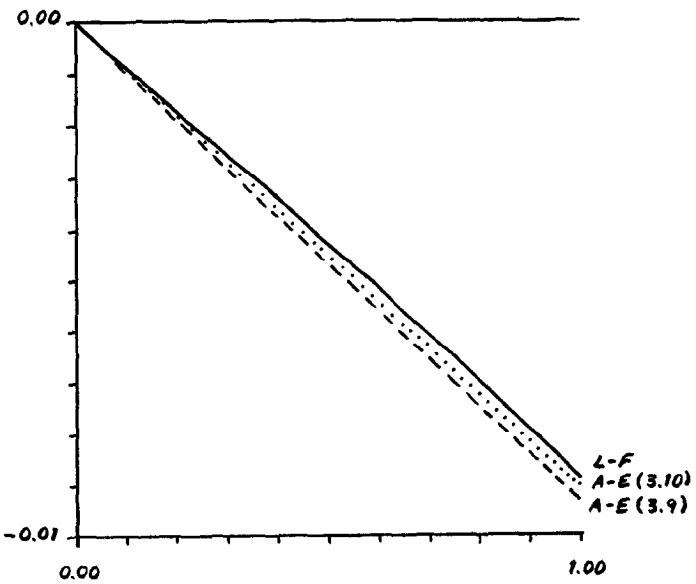


FIG. C3. $v(x, 1)$ for case C. L–F, A–E (3,9), and A–E (3.10).

These values should be compared to the "exact" solution produced by the leap-frog scheme

$$\alpha(1) = 0.99523.$$

In the figures we have not distinguished between A-E-a and A-E-d since they are in all cases very close to each other.

The following table shows the normalized computing time for the three methods used.

A-E-d, $\Delta t = 0.05$	1
C-N, full system, $\Delta t = 0.05$	15
L-F, full system, $\Delta t \approx 0.004$	12

No attempt was made in either case to optimize the codes.

D

$$\rho_0 = 1.67 \cdot 10^{-9} \qquad (\epsilon = 0.0062985);$$

$$g(t) = g_0[1.05 + 0.05 \sin(2\pi t - \pi/2)] \qquad \text{for } t \leqslant 0.5,$$

$$= 1.1\, g_0 \qquad \text{for } 0.5 \leqslant t \leqslant 1;$$

$$v(x, 0) = 0;$$

$$\tilde{B}(x, 0) = [1 - (0.9 + 0.1 \cos 2\pi x)^{\gamma}]\, \epsilon/\gamma M^2.$$

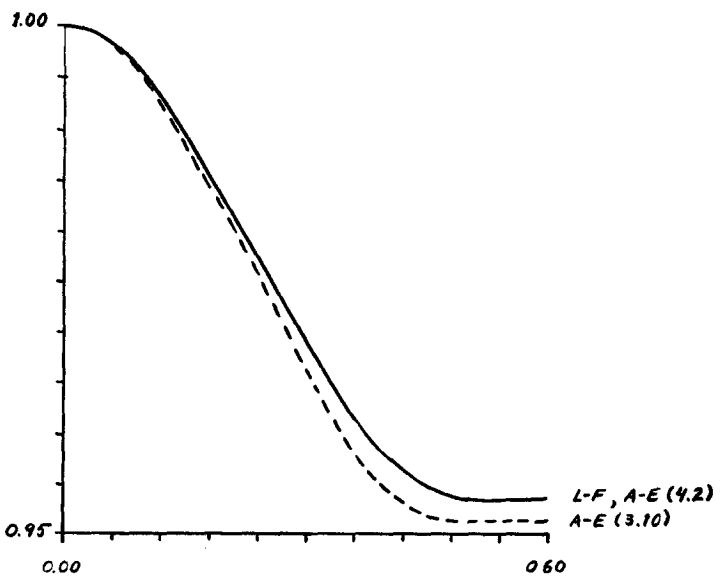


FIG. D1. $\alpha(t)$ for case D. L–F, A–E (3.10), and A–E (4.2).

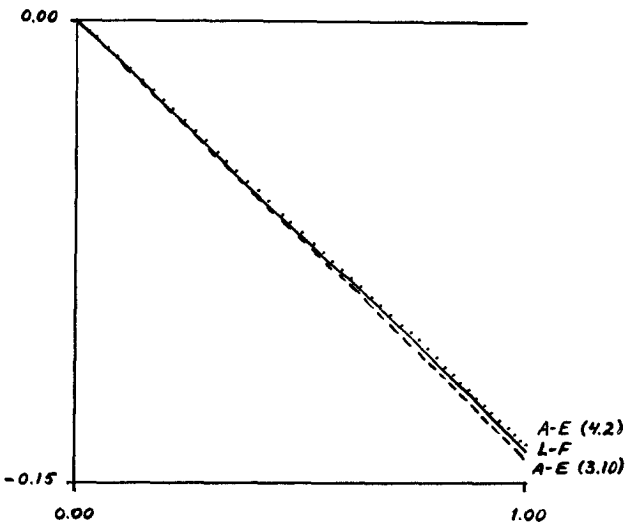


FIG. D2. $v(x, 0.3)$ for case D. L–F, A–E (3.10), and A–E (4.2).

In this case $g''(t)$ is of the order ϵ^{-1}, and we must expect that the ϵ^2-error obtained with (3.9) is reduced to an error of order ϵ. This might still be good enough since ϵ is small, and the result is shown in Figs. D1 and D2.

For this problem, the variable substitution $B = 1 + \epsilon\tilde{B}$ should be changed to

$$B = \frac{g^{1/2}}{B_0} + \epsilon\tilde{B}$$

(which could have been used also for the previous problems). In this way, we obtain a modified version of the third equation, which will now contain an inhomogeneous t-dependent term. We do not write down all the formulas here, but the result for v and α, using the first two terms in the expansions, is

$$\begin{aligned} v(x, t) &= -\frac{g'(t)(g(0))^{1/2}}{2g(t)^{3/2}}\, x, \\ \alpha(t) &= (g(0)/g(t))^{1/2}. \end{aligned} \tag{4.2}$$

These formulas give very accurate results, and for α the curve cannot be distinguished from the true one in Fig. D2.

E

This experiment used the same data as C, but a forcing function

$$F = \left(0, \frac{1}{\epsilon}(1 - e^{0.5x^2}), \frac{1}{\epsilon}e^{x^2}\right)^T$$

was introduced. The true solution and the one obtained by (4.2) are shown in Fig. E1.

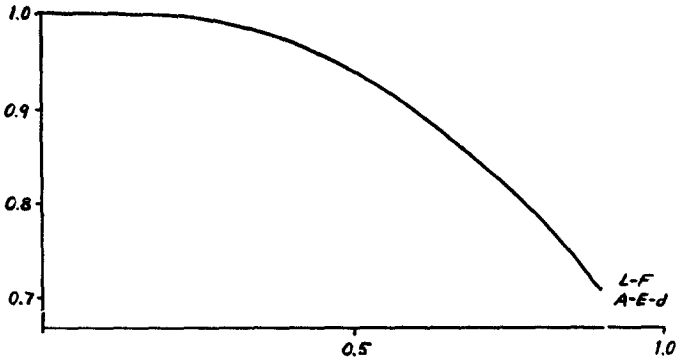


FIG. E1. $\alpha(t)$ for case E. L–F and A–E–d.

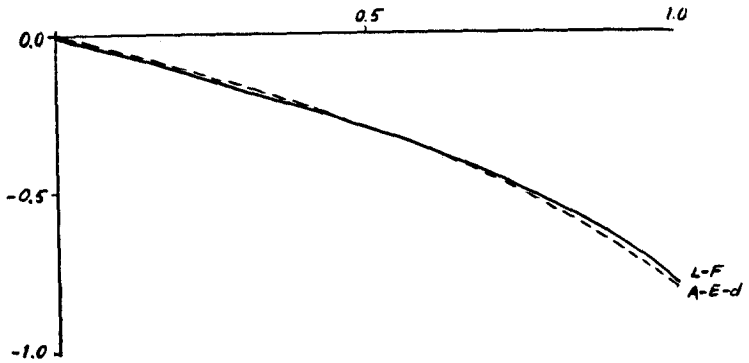


FIG. E2. $v(x, 0.9)$ for case E. L–F and A–E–d.

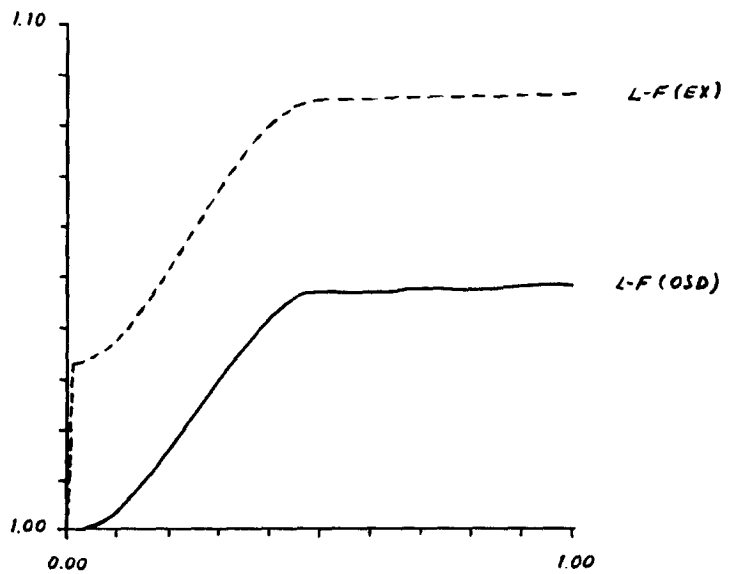


FIG. F1. $\rho(1, t)$ for case C. Leap-frog scheme using extrapolated values at $x = 1$(L–F(EX)) and one-sided differences at $x = 1$(L–F(OSD)).

5. Concluding Remarks

The ideas put forward in this paper can be summarized as follows. If for a certain problem one is interested in the slow variations of the solution only, then the rapidly varying part should not be present, i.e., the solution should be smooth. This is the condition under which implicit schemes can be used with long time steps and it requires that the initial values be chosen properly. However, the smoothness property of the solutions should be used in the form of asymptotic expansions, such that a more efficient solution method is obtained as outlined in this paper. For systems in one space dimension we think that it is quite clear that this method in most cases is superior. In the multidimensional case the elliptic subproblems which must be solved in each time step are more complicated. However, these systems are smaller than the complete system, therefore we can still expect an efficient method. We believe that for a given large system where high efficiency is required, the method arising from the use of asymptotic expansions should be used, and each step be carefully analyzed such that one can make use of all possible simplifications. The asymptotic expansions could also be used in order to solve the equations arising from a semi-implicit scheme in an effective way. We will further investigate these matters in a forthcoming paper.

We also want to stress the point that the equations to be solved in our method do not have any $1/\epsilon$-factors in the coefficients. If the full system is solved, these factors can cause severe numerical difficulties, since truncation errors and rounding errors are affected by these large factors. Furthermore no filtering is required, the method automatically chooses the slow time scale even for nonlinear problems.

Acknowledgment

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